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HYPERBOLIC SYSTEMS OF CONSERVATION LAWS WITH SOME SPECIAL INVARIANCE PROPERTIES

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ABSTRACT
The Euler equations of fluid dynamics are an example of a very special class of nonlinear, hyperbolic systems of conservation laws, in particular those satisfying conditions of reflection and Galilean invariance. These invariance properties are directly responsible for several of the attractive structural features of this system.

I. Introduction

The Euler equations of compressible fluid dynamics are well-known and especially important examples of nonlinear hyperbolic systems of conservation laws. Several features of these systems, not expected in general – the existence of entropy functions, global Riemann invariants, the linearly degenerate "middle field", for example – greatly facilitate the analysis of the local structure of weak solutions of these systems.

These features are not simply fortunate accidents; we shall see here that they arise directly from the Galilean and reflection invariance properties of these systems, and hold also for other systems with these properties. For the Euler equations, these invariance principles are expected on physical grounds; however, in the context of nonlinear hyperbolic systems of conservation laws, they are quite special. Indeed, for systems of dimension three or greater in one space variable, we shall see here that there are essentially only six such systems, two of which correspond to variations of the Euler equations. These systems can be extended to several space dimensions, but not all of the systems obtained permit

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rotational invariance in doing so. Thus the Euler equations are, in fact, very special examples of nonlinear hyperbolic systems.

For obvious reasons, our discussion is centered on the case of a system of dimension three in only one space variable, and proceeds as follows: in the following section, we introduce notation and prove a representation theorem. Hyperbolicity, the linearly degenerate second field, and the existence of entropy functions are discussed in section 3. In section 4, we discuss the shock curves in the large, the entropy condition, the solution of Riemann problems and the stability of discontinuities under perturbation of the incoming characteristics.

For additional background material, we refer the reader to]10] as a general reference and [9] for a thorough discussion of the solution of Riemann problems for the Euler equations.

II. Representation Theorem

We consider systems of the form

$$w_t + f_x = 0 \tag{2.1}$$

where x, t are the scalar independent variables, $w(x,t), f(w) \in \mathbb{R}^n, f$ a given smooth vector valued function of w, with the eigenvalues of f_w real. In place of the components of w, it is convenient to introduce new dependent variables $v = (u, z)^T, u(x,t) \in \mathbb{R}, z(x,t) \in \mathbb{R}^{n-1}$; here u is interpreted as the fluid velocity and z the "internal" variables, for example the mass and internal energy densities. We assume w a smooth function of v, at least locally invertible, i.e. the matrix w_v nonsingular.

Our systems are invariant under reflection, i.e. under the transformation

$$x \to -x, t \to t, u \to -u, z \to z \tag{2.2}$$

and under a Galilean transformation, of the form

$$x \to x - \eta t, t \to t, u \to u - \eta, z \to z \tag{2.3}$$

for any constant η. For a given system, this will not determine z uniquely – an explicit choice is made below. Also we mean invariant here in a strong sense, that under a transformation (2.2) or (2.3), each equation in (2.1) is a linear combination of the equations composing (2.1), with coefficients that may depend on η in the case of (2.3) but do not depend on the dependent variables.

We rewrite (2.1) in the form

$$w(u,z)_t + (uw(u,z) + q(u,z))_x = 0 \tag{2.4}$$

thus determining the vector function q.

In view of (2.2), (2.4) is equivalent to a system of dimension $2n$,

$$(w(u,z) \pm w(-u,z))_t + (u(w(u,z) \pm w(-u,z)) + (q(u,z) \mp q(-u,z)))_x = 0,$$

in which each term is even or odd under the transformation (2.2). Taking a constant, nontrivial linear combination of the equations in (2.4) if necessary, it is therefore no loss of generality to assume that each equation in (2.4) is either even (unchanged) or odd (all terms change sign) under the transformation (2.2), and therefore that the system (2.4) assumes the form

$$w_+(u,z)_t + (uw_+(u,z) + q_-(u,z))_x = 0, \tag{2.5a}$$

$$w_-(u,z)_t + (uw_-(u,z) + q_+(u,z))_x = 0, \tag{2.5b}$$

where $\pm$ subscripts denote even or odd terms under the transformation (2.2). Here w_+, q_- are vectors of dimension $n-m$ and w_-, q_+ vectors of dimension m, for some integer m.

The transformation (2.3) puts the system (2.4) into the form

$$w(u-\eta,z)_t + (uw(u-\eta,z) + q(u-\eta,z))_x = 0 \tag{2.6}$$

where ∂_t is now with $x - \eta t$ fixed. For this to be the same system as (2.4), there must be a nonsingular $n \times n$ matrix $\kappa = \{\kappa_{ij}(\eta)\}$ and n-vectors $\tilde{\kappa}(\eta), \hat{\kappa}(\eta)$ such that

$$w(u-\eta,z) = \kappa(\eta)w(u,z) + \tilde{\kappa}(\eta), \tag{2.7}$$

$$q(u-\eta,z) = \kappa(\eta)q(u,z) - u\tilde{\kappa}(\eta) + \hat{\kappa}(\eta). \tag{2.8}$$

As w and q are smooth functions of u, it follows that $\kappa, \tilde{\kappa}, \hat{\kappa}$ are smooth in η and satisfy $\kappa =$ identity, $\tilde{\kappa} = \hat{\kappa} = 0$ at $\eta = 0$.

Passing to the limit as $\eta \to 0$, we obtain the existence of a constant (likely singular) matrix $\beta = \{\beta_{ij}\}$ and constant vectors $\tilde{\beta}, \hat{\beta}$ such that

$$w_u(u,z) = \beta w(u,z) + \tilde{\beta}, \tag{2.9}$$

$$q_u(u,z) = \beta q(u,z) - \tilde{\beta}u + \hat{\beta}. \tag{2.10}$$

Clearly $\beta_{ij} = 0$ unless w_i and w_j are of opposite parity with respect to the transformation (2.2). For w_i even, $\tilde{\beta}_i$ vanishes, and for w_i odd (and thus q_i even, from (2.5)), $\hat{\beta}_i$ vanishes.

Next we show

LEMMA 2.1: *The dimension of w_- and of q_+ is 1, independent of n.*

Proof: Let m be the dimension of w_- and of q_+, and thus $n-m$ that of w_+ and of q_-. We require $w(u,z)$ to be locally invertible. At a point where $u=0, w_-$ vanishes and we shall have the $n-m$ components of w_+ determined by the $n-1$ components of z, plus $u=0$. This cannot be locally invertible if m exceeds one.

And if $m=0$, i.e. all the equations are even, at least one of the components w_i must depend nontrivially on u; otherwise u cannot be determined from w. Then $\partial w_i/\partial u$ does not vanish identically, but $\tilde{\beta}_i=0$ as w_i is even, so there exists some j such that $\beta_{ij}\neq 0$. But then w_j is odd, so there must be an odd equation. ☐

Some choice for the $n-1$-vector function z is implicit in (2.2), (2.3), but this does not determine z uniquely. Given some choice for z such that (2.2), (2.3) are satisfied, set $\tilde{z}=w_+(0,z)$, which by Lemma 2.1 is a vector function of z, of dimension $n-1$. And as $w_{-z}(0,z)$ vanishes, for $w_v(0,z)$ to be nonsingular, the matrix $\tilde{z}_z$ must be nonsingular. Thus variations of w,q as u varies with z fixed are equivalent to variations as u varies, $\tilde{z}$ fixed, and (2.2), (2.3) hold as well with z replaced by $\tilde{z}$. Regarding (2.9) as an initial value problem for w, with u as the independent variable, recalling $w_-(0,z)=0$, it is clear that $u,\tilde{z}$ uniquely determine $w(u,z(\tilde{z}))$ and that $u, w(u,z(\tilde{z}))$ uniquely determine $w(0,z(\tilde{z}))$. Thus $\tilde{z}=w_+(0,z)$ satisfies $\tilde{z}=w_+(0,z(\tilde{z}))$. The global invertibility of the transformation $u,\tilde{z}\to w(u,z(\tilde{z}))$ thus depends only on obtaining u uniquely in the large, given $w(u,z(\tilde{z}))$.

Therefore hereafter we identify

$$z=w_+(0,z) \tag{2.11}$$

without loss of generality in (2.2), (2.3).

With these preliminaries, we have the following representation theorem.

THEOREM 2.2: *Let (2.1) be a system of dimension 3, invariant under the transformation (2.2) and (2.3). Then for some equation of state $P=P(\rho,e)$ and some value of a scalar constant α, it is one of the following six systems, possibly after a scaling in u, and the appropriate choice of ρ, e, possibly involving scaling and the absorption of additive constants.*

$$\begin{aligned} &\rho_t=(\rho u+\alpha u)_x=0\\ &(\rho u)_t+(\rho u^2+P+\frac{\alpha}{2}u^2)_x=0\\ &e_t+(eu)_x=0 \end{aligned} \tag{2.12a}$$

$$\begin{aligned} &\rho_t + (\rho u + \alpha u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + P + \frac{\alpha}{2}u^2)_x = 0 \\ &(e + \frac{1}{2}\rho u^2)_t + (u(e + \frac{1}{2}\rho u^2 + P) + \frac{\alpha}{6}u^3)_x = 0 \end{aligned} \tag{2.12b}$$

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(e \sinh u)_t + (ue \sinh u + P \cosh u)_x = 0 \\ &(e \cosh u)_t + (ue \cosh u + P \sinh u)_x = 0 \end{aligned} \tag{2.12c}$$

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(e \sin u)_t + (ue \sin u - P \cos u)_x = 0 \\ &(e \cos u)_t + (ue \cos u + P \sin u)_x = 0 \end{aligned} \tag{2.12d}$$

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + (\frac{u^2}{2} + P)_x = 0 \\ &e_t + (ue)_x = 0 \end{aligned} \tag{2.12e}$$

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + (\frac{u^2}{2} + P)_x = 0 \\ &(e + \frac{1}{2}u^2)_t + (\frac{u^3}{3} + u(e + P))_x = 0 \end{aligned} \tag{2.12f}$$

Proof: We consider the constants $\beta, \tilde{\beta}, \hat{\beta}$ in (2.9), (2.10). Now w_{-u} cannot vanish identically; if w_{-u} is not simply a constant, by taking a linear combination of the w_+ as necessary, we can make $w_{-u} = w_i$ for some i. As each component of w_{+u} is either zero or a multiple of w_-, either $w_{i,u} = 0$ or $w_{i,u} = \nu w_-$ for some nonzero ν. Let w_j be the other component of w_+.

If $w_{i,u} = 0$, then either $w_{j,u} = 0$, leading to the system (2.12a), or else, possibly after a scaling, $w_{j,u} = w_-$, giving (2.12b).

If $w_{i,u} = \nu w_-, \nu \neq 0$, then possibly after a scaling in u, $|\nu| = 1$; taking a linear combination of w_i and w_j if necessary, we can have $w_{j,u} = 0$. The value $\nu = +1$ gives (2.12c); $\nu = -1$ gives (2.12d).

And if w_{-u} is a constant, the constant can be taken as $+1$. By taking linear combinations of the components of w_+, again denoted by w_i, w_j, we can have $w_{i,u} = 0$ and either $w_{j,u} = 0$ or else $w_{j,u} = w_-$. These are the two systems (2.12e) and (2.12f). □

In each case (2.9), (2.10) are used to obtain the appropriate expression for q, with P the "constant of integration" in the odd equation. The resulting system is obtained from (2.4).

Entirely similar arguments apply to systems of higher dimension, and show that one obtains the same systems (2.12) with additional "continuity equations", of the form $\mu_{i,t} + (u\mu_i + \alpha_i u)_x = 0, i = 1, \ldots$. In this case P depends in general also on the μ_i.

Extensions to higher space dimensions are also possible. These are well-known for the Euler equations. For example, an extension of (2.12c) to two space dimensions, satisfying the obvious extension of (2.2) and (2.3), in given by

$$
\begin{aligned}
&\rho_t + (\rho u_1)_x + (\rho u_2)_y = 0\\
&(e_1 \sinh u_1)_t + (u_1 e_1 \sinh u_1 + P \cosh u_1)_x + (u_2 e_1 \sinh u_1)_y = 0\\
&(e_1 \cosh u_1)_t + (u_1 e_1 \cosh u_1 + P \sinh u_1)_x + (u_2 e_1 \cosh u_1)_y = 0\\
&(e_2 \sinh u_2)_t + (u_1 e_2 \sinh u_2)_x + (u_2 e_2 \sinh u_2 + P \cosh u_2)_y = 0\\
&(e_2 \cosh u_2)_t + (u_1 e_2 \cosh u_2)_x + (u_2 e_2 \cosh u_2 + P \sinh u_2)_y = 0
\end{aligned}
\tag{2.13}
$$

in the five unknowns u_1, u_2, ρ, e_1, e_2, with $P = P(\rho, e_1, e_2)$.

The systems (2.12a), (2.12b) can be interpreted, obviously, as modest extensions of the Euler equations, with e interpreted as the entropy density for isentropic flow, or as a perpendicular momentum component rather than the internal energy density in (2.12a). The systems (2.12e), (2.12f) may be viewed as extensions of Burgers equation. (2.12c) appears attractive from the point of view of local structure of weak solutions, but we are unaware of any applications or appearances of this system. As against that, there is trouble with (2.12d); the components of w do not uniquely determine u, and this will be reflected in trouble with the entropy functions and with the shock curves.

One notes that the equation of fluid dynamics in Lagrangian coordinates does not appear in (2.12). Because of the different interpretation of $\partial/\partial t$ in Lagrangian coordinates, the Lagrangian system does not satisfy (2.3). Similarly, the Lundquist equations for ideal magnetofluid dynamics (in one space variable) satisfy (2.3), but satisfy (2.2) only in the special case that the (constant) longitudinal component of the magnetic field is equal to zero. In this case the Lundquist system reduces to the Euler system with additional continuity equations and different expressions for pressure and energy density.

III. Characteristic Speeds and Entropy Functions

Hereafter we assume $n \geq 3$ for definiteness and simplicity.

It is convenient to discuss the characteristic speeds λ_i and corresponding eigenvectors r_i, $i = 1, \ldots, n$, in the v-coordinates, $v = (u, z)^T$, i.e. the solutions of

$$(f_v - \lambda_i w_v) r_i = 0, \quad i = 1, \ldots, n. \tag{3.1}$$

The representation (2.5) puts (3.1) into a more useful block form

$$\left[\begin{pmatrix} w_+ + q_{-u} & q_{-z} \\ w_- + q_{+u} & q_{+z} \end{pmatrix} - (\lambda_i - u) \begin{pmatrix} w_{+u} & w_{+z} \\ w_{-u} & w_{-z} \end{pmatrix} \right] r_i = 0, \quad i = 1, \ldots, n. \tag{3.2}$$

From (2.2) and (2.3), it can be inferred that $\lambda_i = u + c_i$, where the c_i are independent of u and satisfy $c_i = -c_{n+1-i}$, and that the r_i are independent of u. Therefore we can determine the c_i, r_i with $u = 0$ and the vanishing of all the odd terms in (3.2).

One solution is $\lambda_2 = u$, i.e. $c_2 = 0$, with corresponding eigenvector

$$r_2 = (0, \tau)^T, \quad \tau(z) \in \mathbf{R}^{n-1} \tag{3.3}$$

satisfying

$$q_{+z}(0, z) \cdot \tau(z) = 0; \tag{3.4}$$

$q_+(0, z)$ is a scalar function independent of u, equal to P in each of the systems (2.12). Without loss of generality, we will continue this identification hereafter. Therefore there are $n - 2$ independent nontrivial choices for τ, and λ_2 is of multiplicity (at least) $n - 2$. Since $u_v = (1, 0)^T$ in these coordinates, we have from (3.3)

$$r_2 \cdot \lambda_{2,v} = 0, \tag{3.5}$$

i.e. all of the fields corresponding to λ_2 are linearly degenerate.

Thus $u = \lambda_2$ is clearly one Riemann invariant for each of these fields, and from (3.4), $q_+ = P$ is another 2-Riemann invariant. Indeed, from (3.2) and (3.3), every component of the vector q is a 2-Riemann invariant, but these are all functions of u and P.

Making an ansatz for λ_n, λ_1 of the form

$$\lambda = u \pm c, \quad r = \begin{pmatrix} 1 \\ \pm\varsigma \end{pmatrix}$$

in (3.2) and again setting $u = 0$, we readily find

$$w_+(0, z) + q_{-u}(0, z) - c\varsigma(z) = 0 \tag{3.7}$$

and

$$\varsigma(z)\cdot q_{+z}(0,z)-cw_{-u}(0,z)=0 \tag{3.8}$$

so that

$$\begin{aligned} c^2 &= \frac{q_{+z}(0,z)\cdot(w_+(0,z)+q_{-u}(0,z))}{w_{-u}(0,z)} \\ &= P_z\cdot(z+q_{-u}(0,z))/w_{-u}(0,z) \end{aligned} \tag{3.9}$$

using (2.11), and

$$\varsigma(z)=(z+q_{-u}(0,z))/c(z). \tag{3.10}$$

For the systems (2.12), we set $z=(\rho,e)^T$; $q_{-u}(0,z)$ vanishes for (2.12a) and (2.12e), and is equal to $(0,P)^T$ otherwise; and $w_{-u}(0,z)$ is equal to ρ for (2.12a), (2.12b), e for (2.12c), (2.12d), and 1 for (2.12e), (2.12f).

Hyperbolicity thus follows for the right side of (3.9) nonnegative, but strict hyperbolicity can hold only for $n\le 3$, with the right side of (3.9) positive. Furthermore, strict hyperbolicity cannot be maintained in a neighborhood of a point where $w_{-u}(0,z)=0$. For if $|c|$ remains bounded in such a neighborhood, $\varsigma(z)\cdot P_z\to 0$ as $|w_{-u}(0,z)|\to 0$, so ς approaches a linear combination of the vectors r and the eigenvectors r_1,r_2,r_3 cease to be linearly independent.

For $\lambda=\lambda_n=u+c, c>0, r=r_n=(1,\varsigma)^T$, any value of u, the even and odd parts of (3.2) have to be individually satisfied, so in particular

$$q_{+z}\cdot\varsigma=cw_{-u} \tag{3.11}$$

and

$$q_{-z}\cdot\varsigma=cw_{+u}, \tag{3.12}$$

so

$$q_z\cdot\varsigma=cw_u. \tag{3.13}$$

We use this to prove

LEMMA 3.1: *Assume $c > 0$; then the $(n-1) \times (n-1)$ matrix $z_w q_x$ vanishes identically.*

Proof: The components of q are 2-Riemann invariants, so $q_x \tau = 0$ for any τ satisfying (3.4). Furthermore, using (3.13)

$$\begin{aligned} z_w q_x \varsigma &= c z_w w_u \\ &= c z_u \\ &\equiv 0; \end{aligned}$$

as ς and the τ span $\mathbb{R}^{n-1}$, for $c > 0$ and thus the eigenvectors r_i necessarily independent, our result follows. □

This lemma is used in the determination of entropy functions for the systems (2.4). For simplicity, we shall restrict attention to the case $n = 3$ in the discussion of entropy functions. An entropy function/flux is a pair of scalar functions (of w, say) U, F, such that continuous solutions of (2.4) satisfy an additional conservation law (cf. [2])

$$U(w)_t + F(w)_x = 0. \tag{3.14}$$

Trivial examples of (3.14) are U constant and $U = a \cdot w$ for some constant vector a. More interesting examples correspond to U strictly convex in w, or at least U_{ww} nonsingular.

Not surprisingly, entropy functions for systems (2.4), invariant under the transformation (2.2), (2.3), necessarily satisfy some special conditions.

LEMMA 3.2: *Let U, F be an entropy function/flux pair for a system (2.1) or (2.4), invariant under (2.2) and (2.3). Then*

(i) U, F can be taken of opposite parity with respect to (2.2).

(ii) $Q \equiv F - uU$ is a 2-Riemann invariant, indeed a k- Riemann invariant for all k such that $\lambda_k = u$ (for $n > 3$).

(iii) $U_u = U_w w_u$ is also an entropy function for (2.4), with corresponding flux $F_u - U = uU_u + Q_u$.

Proof: The necessary and sufficient condition for U, F to be an entropy function/flux pair is that

$$U_w f_w = F_w; \tag{3.15}$$

assuming (3.15) holds, (3.14) is obtained by taking the scalar product of (2.1) with U_w. Therefore (3.14) will also be invariant under (2.2) and (2.3). In particular, the even and odd parts of (3.14) under (2.2) will have to be separately satisfied, so the first statement holds. For any constant η, making the transformation (2.3) in (3.14), we find

$$U(w(u-\eta,z))_t + \eta(U(w(u-\eta,z))_x + F(w(u-\eta,z))_x = 0; \tag{3.16}$$

taking the limit as $\eta \to 0$, we obtain the third statement of the lemma. The second statement is simply a statement that the "entropy drop" $-s[U] + [F]$ vanishes for a linearly degenerate field, and here we have $s = \lambda_k = u$. Here $[\ ]$ denotes the jump in the enclosed quantity at a discontinuity satisfying the Rankine-Hugoniot conditions, as usual. □

Lemma 3.2 facilitates the search for entropy functions. An obvious possibility, already well-known in the context of the Euler equations, is to take $U = U(z)$, independent of u, and $F = uU$. In particular, for U, F of this special form the statements (ii) and (iii) of Lemma 3.2 are trivially satisfied.

For the systems (2.12a), (2.12e), there is an additional entropy function corresponding to an energy–these are discussed below. Otherwise, partial results suggest the absence of other entropy functions. For example, for the Euler system (2.12b) it is known that in general no other nontrivial entropy functions exist [11].

We thus seek entropy functions of the special form

$$U(z) = (\rho + \alpha)\sigma(z), \quad F(u,z) = uU(z) = u(\rho+\alpha)\sigma(z) \tag{3.17}$$

where ρ is the first component of w and of z, always satisfying the "continuity equation", as in each of the examples (2.12). By convention, $\alpha \equiv 0$ for the systems (2.12) other than (2.12a), (2.12b).

Then we compare

$$F_x = uU_z z_x + U u_x \tag{3.18}$$

with F_x as determined from the requirement that U, F are an entropy function/flux pair, i.e. (3.15), leading to

$$\begin{aligned} F_x &= U_w f_x \\ &= U_w (uw + q)_x \\ &= U_z z_w (w u_x + u w_x + q_u u_x + q_z z_x) \\ &= U_z z_w u w_x + U_z z_w (w + q_u) u_x + U_z z_w q_z z_x \\ &= uU_x + U_z z_w (w + q_u) u_x \end{aligned} \tag{3.19}$$

using Lemma 3.1 in the last step, tacitly assuming $c > 0$.

We can simplify (3.19) by setting $\lambda_i = u + c, r_i = (1, \varsigma)^T$ in (3.2), obtaining

$$w + q_u + q_z\varsigma - cw_u - cw_z\varsigma = 0. \tag{3.20}$$

Multiplying (3.20) by z_w and using Lemma 3.1 again, we find

$$\begin{aligned} z_w(w + q_u) &= c\varsigma \\ &= z + q_{-u}(0, z) \end{aligned} \tag{3.21}$$

using (3.10), so $z_w(w+q_u)$ is in fact independent of u. Comparing (3.18), (3.19), (3.21), we have

$$U = U_z(z + q_{-u}(0, z)); \tag{3.22}$$

using (3.17) and (3.10) again, we have

LEMMA 3.3: *Assume $c^2 > 0$ (as determined by (3.9)); then a necessary and sufficient condition that U, F, of the form (3.17), be an entropy function/flux pair is that*

$$\varsigma \cdot \sigma_z = 0, \tag{3.23}$$

i.e. the specific entropy σ is both a 1 - and an n-Riemann invariant.

Let e be the second component of z, such that $w_{-u}(0, z)$ is either ρ, e, or 1. We restrict attention to z satisfying

$$\rho > 0, \quad \rho + \alpha > 0, \quad e > 0, \tag{3.24}$$

and assume that $c^2 > 0$ as determined from (3.9). We wish to establish the global solvability of (3.23). In the region determined by (3.24), there are no critical points of the system

$$z' = \varsigma(z); \tag{3.25}$$

both ρ and P are Lyapunov functions for this system, so there are no bounded semiorbits.

For the system (2.12a), (2.12e), i.e. those systems not containing an "energy equation", $\varsigma(z) = (z + (\alpha, 0, 0, \ldots))/c(z)$, so the orbits of (3.25) are just straight lines in the variables $\rho+\alpha, e$ and the solvability of (3.23) is obvious. For the other systems, the second component of ς is proportional to $e + P$, and the question of solvability is more interesting. We prove the following lemma:

LEMMA 3.4: *For the systems (2.12b), (2.12c), (2.12d), (2.12f), assume $c(z) > 0$ in a region $\Omega : \rho - \rho_0 \geq \varepsilon_1 > 0, 0 < \varepsilon_2 \leq e \leq M$, where $\rho_0 =$ maximum $(0, -\alpha)$ and $\varepsilon_1, \varepsilon_2, M$ are given. Then σ is determined uniquely in this region by (3.23) and its values on the line segments $\rho = \rho_0 + \varepsilon_1, (e = \varepsilon_2) \cap (e + P > 0)$, and $(e = M) \cap (e + P < 0)$.*

Proof: From (3.25), (3.9) and (3.10) we have in such a region

$$\rho' = \frac{\rho + \alpha}{c} \geq \frac{\varepsilon_1}{c} > 0, \tag{3.26}$$

$$e' = \frac{e + P}{c}, \tag{3.27}$$

$$\begin{aligned}(e + P)' &= \frac{e + P}{c} + c w_{-u}(0, z) \\ &> \frac{e + P}{c}\end{aligned} \tag{3.28}$$

(prime denoting differentiation along the orbits of (3.25)), so the characteristics are entering the region Ω on each of the prescribed boundary surfaces. If $e = M, \quad e + P = 0$, (3.27), (3.28) shows that e increases immediately so the orbit leaves $\bar{\Omega}$. If $e = \varepsilon_2, \quad e + P \leq 0$, and $\rho \geq \rho_0 + \varepsilon_1$, then along the orbit, in the negative direction, ρ decreases and $e + P$ remains negative (or immediately becomes and remains negative) so e increases. Continuing in this direction, using (3.26) - (3.28) it follows that such an orbit will reach either the line $\rho = \rho_0 + \varepsilon_1$, or $e = M$, with $e + P < 0$.

An orbit from a point $\rho = \rho_0 + \varepsilon_1, \quad \varepsilon_2 < e < M$ or a point $\rho \geq \rho_0 + \varepsilon_1, \quad e = M$ cannot reach $e = \varepsilon_2, \quad \rho > \rho_0 + \varepsilon_1$ with $e' = (e + P)/c$ positive, nor can it reach $e = M, \quad \rho > \rho_0 + \varepsilon_1$, with e' negative. So the boundary conditions are consistent.

It is clear from (3.26) - (3.28) that the orbit backward from any interior point in Ω will reach one of the boundary segments on which σ is prescribed. Thus (3.23) and the given boundary values uniquely determine σ throughout Ω. □

Next we establish the existence of such entropy functions which are convex in z. For simplicity, we write out the proof only for the systems (2.12b), (2.12c), (2.12d), (2.12f); the other cases are simpler.

LEMMA 3.5: *Under the conditions of Lemma 3.4, for any given $K > 0$ we can prescribe σ on the specified boundary segments such that σ_e is continuous,*

$$\sigma_{ee} \geq K|\sigma_e|, \quad \sigma_e < 0 \tag{3.29}$$

uniformly in a neighborhood of the line segments where σ is prescribed.

Proof: Writing (3.23) in the form

$$(\rho+\alpha)\sigma_\rho + (e+P)\sigma_e = 0 \tag{3.30}$$

and taking ρ, e derivatives we obtain

$$(\rho+\alpha)\sigma_{\rho\rho} + (e+P)\sigma_{\rho e} + \sigma_\rho + P_\rho\sigma_e = 0, \tag{3.31}$$

$$(\rho+\alpha)\sigma_{\rho e} + (e+P)\sigma_{ee} + (1+P_e)\sigma_e = 0, \tag{3.32}$$

$$(\rho+\alpha)\sigma_{\rho ee} + (e+P)\sigma_{eee} + 2(1+P_e)\sigma_{ee} + P_{ee}\sigma_e = 0, \tag{3.33}$$

so that along orbits of (3.25), we have from (3.32), (3.33)

$$\sigma_e' = -(1+P_e)\sigma_e/c, \tag{3.34}$$

$$\sigma_{ee}' = -(2(1+P_e)\sigma_{ee} + P_{ee}\sigma_e)/c. \tag{3.35}$$

Clearly we can prescribe σ_e, σ_{ee} satisfying (3.29) on the segment $\rho = \rho_0 + \varepsilon_1, \varepsilon_2 < e < M$.

At a point where $e + P = 0$, $\rho + \rho_0 \geq 0$, $c^2 > 0$ in (3.9) requires $P_\rho > 0$ so $(e+P)_\rho > 0$. Thus in each of the boundary segments $e = \varepsilon_2$ or $e+M$, $e+P < 0$ at most in a single interval $e = \varepsilon_2$, $\varepsilon_1 \leq \rho < \rho_-$ or $e = M$, $\varepsilon_1 \leq \rho < \rho_+$. First we consider an interval $\varepsilon_1 \leq \rho < \rho_+$, $e = M$. Choosing $\sigma_e(\rho_+, M) < 0$ arbitrarily, we have from (3.32), for σ_{ee} chosen satisfying (3.29),

$$\begin{aligned}(\rho+\alpha)\sigma_{\rho e} &= -(1+P_e)\sigma_e - (e+P)\sigma_{ee} \\ &\geq -(1+P_e)\sigma_e - K|e+P|\sigma_e \\ &\geq -(\text{constant})\sigma_e\end{aligned} \tag{3.36}$$

since $e + P \leq 0$, so as ρ decreases from ρ_+ to ε_1, σ_e remains negative. Thus $\sigma_e(\varepsilon_1, M) < 0$ is determined; we prescribe σ_e, σ_{ee} on the segment $\rho = \rho_0 + \varepsilon_1$, $\varepsilon_2 < e < M$, satisfying (3.29), thus obtaining $\sigma_e(\varepsilon_1, \varepsilon_2) < 0$.

Now on the segment $\varepsilon_1 \leq \rho \leq \rho_-$, $e = \varepsilon_2$, the values of σ_e are determined from the boundary values so far obtained and (3.34), since the orbits through these points continue backward to the other specified boundary segments. Thus $\sigma_e(\rho_-, \varepsilon_2) < 0$ is continuously obtained. For $\rho > \rho_-$, $e = \varepsilon_2$, we have $e+P > 0$ and so $\sigma_{\rho e} \leq -$ (constant) σ_e from (3.36). Thus for σ_{ee} chosen satisfying (3.29), σ_e remains negative, as ρ increases, and the proof is complete. □

As a corollary of Lemmas 3.3, 3.4, 3.5, we have the following:

THEOREM 3.6: *Let Ω_0 be bounded within some region Ω where the conditions of Lemma 3.4 hold ($c^2 > 0$ in particular). Then there exists an entropy function/flux pair, of the form (3.17), with U uniformly convex in z within Ω_0.*

Proof: It suffices to show that U_{ee} and $U_{\rho\rho}U_{ee} - U_{\rho e}^2$ can be made positive in Ω_0. A straightforward calculation using (3.17), (3.30), (3.31), (3.32) gives

$$(3.37)\qquad \begin{aligned} U_{\rho\rho}U_{ee} - U_{\rho e}^2 = &- [(e+P)P_e + (\rho+\alpha)P_\rho]\sigma_e\sigma_{ee} \\ &- P_e^2\sigma_e^2 - c^2 w_{-u}(0,z)\sigma_e\sigma_{ee} - P_e^2\sigma_e^2; \end{aligned}$$

since $U_{ee} = (\rho+\alpha)\sigma_{ee}$, it is clearly sufficient that $\sigma_e < 0$ and $\sigma_{ee}/|\sigma_e|$ be sufficiently large. As Ω_0 is bounded, the orbits of (3.25) through any point in Ω_0, continued in the negative direction, reach the prescribed boundary segments in finite time. Therefore using (3.34) and (3.45), choosing K sufficiently large in (3.29) (depending, of course, on Ω_0), these conditions can be achieved. □

Whether such $U(z)$ convex in z is also convex in w, as desired, depends on the specific system (2.12). For (2.12b), (2.12c), and (2.12 f), a calculation shows that if $U = U(z)$ is strictly convex in z then it is also strictly convex in w. For (2.12d) convexity in w fails. For (2.12a), (2.12e), i.e. the systems with no energy equation, $z = w_+$ for all u, so U is independent of w_-, and $U = U(z)$ strictly convex in z is convex, but not strictly convex, in w. However, at least for these two systems, there are other interesting entropy functions.

Indeed, in each of the systems (2.12a), (2.12e) we can identify e as an entropy function of the form (3.17) for the corresponding system (2.12b) or (2.12f). Thus we seek an entropy function for (2.12a) (resp. (2.12e)) in the form of the energy for (2.12b) (resp. 2.12f).

LEMMA 3.7: *Assume that $U = U(h,E)$, $F(h,E)$ are an entropy function/flux pair for a system (2.4), satisfying (2.2), (2.3), rewritten in the form*

$$(3.38)\qquad h_t + b(h,E)_x = 0,$$

$$(3.39)\qquad E_t + H(h,E)_x = 0,$$

with $h, b \in \mathbb{R}^{n-1}$ and E, H scalar valued. Assume that $\partial U/\partial E$ (with h constant) does not vanish. Then $\omega = \omega(h,U)$, $H(h,\omega)$ is an entropy function/flux pair for the system

$$(3.40)\qquad h_t + b(h,\omega)_x = 0,$$

$$U_t + F(h,\omega)_x = 0, \tag{3.41}$$

where ω is the inverse function of U at constant h, i.e.

$$E = \omega(h, U(h,E)). \tag{3.42}$$

Proof: With E identified as $\omega(h,U)$, we wish to show that smooth solutions of (3.40), (3.41) also satisfy (3.39). Since U, F are an entropy function/flux pair, $F = F(h,E)$ satisfies

$$F_x = U_h b_x + U_E H_x, \tag{3.43}$$

and $U = U(h,E)$ satisfies (3.41)

$$\begin{aligned} 0 &= U_t + F_x \\ &= U_h h_t + U_E E_t + F_x \\ &= U_h h_t + U_E E_t + U_h b_x + U_E H_x \\ &= U_E(E_t + H_x) \end{aligned}$$

using (3.40). As U_E is nonzero by assumption, (3.39) holds. □

COROLLARY: *Let U, F be an entropy function/flux pair of the form (3.17) for the system (2.12b) (resp. (2.12f)), with $\sigma_e < 0$ in some region Ω of z-space satisfying (3.24). Let $\omega = \omega(\rho, U)$ be such that $U = U(\rho, e) = U(\rho, \omega(\rho, U))$. Then $\omega + \frac{1}{2}\rho u^2, u(\omega + \frac{1}{2}\rho u^2 + P) + \frac{\alpha}{6}u^3$ (resp. $\omega + \frac{1}{2}u^2, u(\omega + P) + \frac{1}{3}u^3$) is an entropy function/flux pair for (2.12a) (resp. (2.12e)), in the same region Ω.*

Set $\tilde{U} = \omega + \frac{1}{2}\rho u^2$ for (2.12a) or $\omega + \frac{1}{2}u^2$ for (2.12e). If $\sigma_e < 0, \quad \sigma_{ee} > 0$ and (3.24) holds, then $\tilde{U}_{w_- w_-} > 0$. Taking a linear combination $\tilde{U} + KU, \quad K$ a sufficiently large positive constant and U uniformly convex in z, as obtained in Theorem 3.6, $\tilde{U} + KU$ will be uniformly convex in w, as desired. Therefore, we have, finally

THEOREM 3.8: *For any system (2.12) except (2.12d), suppose that $c^2 > 0$ (i.e. that the system is strictly hyperbolic) in a region $\Omega : \rho_0 + \varepsilon_1 \le \rho \le L, \quad \varepsilon_2 \le e \le M, \quad \varepsilon_1, \varepsilon_2, \quad L, M > 0$. Then there exists an entropy function/flux pair U, F for this system, with U uniformly convex in w, in the region determined by $z(w) \in \Omega$.*

IV Discontinuities and the Riemann Problem

We assume hereafter a system (2.1) of dimension 3 satisfying (2.2), (2.3), and such that w uniquely determines u, i.e. excluding (2.12d). For simplicity we assume $\alpha = 0$ in (2.12a), (2.12b), and consider each system in a region D of z-space, given by

$$(4.1)\qquad D = \begin{cases} \{\rho > 0, e > 0\} & \text{for (2.12a),(2.12c),(2.12e)} \\ \{\rho > 0, e + P > 0\} & \text{for (2.12b), (2.12f)} \end{cases}$$

identifying the boundary segment $\{\rho = 0\}$ as the vacuum state. For $z \in D$, we assume an equation of state $P = P(\rho, e)$ such that

$$(4.2)\qquad c^2(z) > 0, \quad P(z), P_\rho(z), P_e(z) \geq 0,$$

with c^2 determined from (3.9). We shall also assume that

$$(4.3)\qquad P(\rho, e) \to \infty \quad \text{as both } \rho, e \to \infty.$$

Two states $v_1 = (u_1, z_1)$ and $v_2 = (u_2, z_2)$ can be connected by a discontinuity of speed $s = s(v_1, v_2)$ in a weak solution of (2.1) if the Rankine–Hugoniot relations are satisfied, written here in the form

$$(4.4)\qquad s(w_{1+} - w_{2+}) = u_1 w_{1+} - u_2 w_{2+} + q_{1-} - q_{2-},$$

$$(4.5)\qquad s(w_{1-} - w_{2-}) = u_1 w_{1-} - u_2 w_{2-} + q_{1+} - q_{2+},$$

using (2.5).

For $z_1 \in D$, let $\Gamma(v_1)$ denote the set of points v_2 satisfying (4.4), (4.5). In a neighborhood of v_1, from the strict hyperbolicity condition (4.2), we have [1]

$$(4.6)\qquad \Gamma(v_1) = \bigcup_{k=1}^{n} (\Gamma_k^+(v_1) \cup \Gamma_k^-(v_1))$$

where each $\Gamma_k^\pm(v_1)$ is a smooth one-parameter manifold with v_1 as one endpoint such that $s(v, v_1) \to \lambda_k(v_1)$ and $(v - v_1)/|v - v_1|$ becomes parallel to $r_k(v_1)$ as $v \to v_1$ within $\Gamma_k^\pm(v_1)$.

To discuss the set $\Gamma(v_1)$ in the large, we first appeal to (2.3), from which it follows that (4.4), (4.5) is invariant if the same constant is added to u_1, u_2, s. Without loss of generality we set $u_2 = 0$; the odd terms at u_2 vanish, w_{2+} is z_2, q_{2+} is P_2, and obtain

$$(4.7)\qquad s(w_{1+} - z_2) = u_1 w_{1+} + q_{1-},$$

$$sw_{1-} = u_1 w_{1-} + q_{1+} - P_2. \tag{4.8}$$

It is also convenient to differentiate (4.4), (4.5), along a curve of states connected to v_1, for v_1 fixed. Denoting such derivatives by dots, and again subsequently putting $u_2 = 0$, we find

$$(z_2 + q_{-u}(0, z_2))\dot{u}_2 - s\dot{z}_2 = \dot{s}(z_2 - w_{1+}), \tag{4.9}$$

$$P_z(z_2)\dot{z}_2 - sw_{-u}(0, z_2)\dot{u}_2 = -\dot{s}w_{1-}. \tag{4.10}$$

From (3.2)–(3.4), one set of solutions of (4.4), (4.5) is obtained as the points on the orbits through v_1 of the system

$$\dot{z} = \pm\tau(z), \quad \tau(z) \cdot P_z(z) = 0, \quad |\tau(z)| = 1; \quad \dot{s} = \dot{u} = 0. \tag{4.11}$$

These are the contact discontinuities, corresponding to $\Gamma_2^{\pm}(v_1)$ in (4.6), as $s = u_1 = \lambda_2(v_1)$. The pressure P is constant on these orbits, but the specific entropy σ, satisfying (3.23) with $\sigma_\rho > 0$, $\sigma_e < 0$ in D, is a Lyapunov function on these orbits, given the strict hyperbolicity (4.2), and noting that $w_{-u}(0, z) > 0$ for all $z \in D$. Thus these orbits extend to infinity or to the boundary of D, and $\Gamma_2^+(v_1)$ may be distinguished from $\Gamma_2^-(v_1)$ by $\sigma > \sigma(z_1)$ in the former case.

We characterize these solutions further in the following three lemmas.

LEMMA 4.1: *If either u_1 or s is zero in a solution of (4.7), (4.8), or if $u_1 = s$, then v_1, v_2 are connected by a contact discontinuity, i.e. one of the two semiorbits of (4.11) beginning at v_1 contains v_2.*

Proof: If $s = 0$, then $u_1 = 0$ follows from (4.7), using (2.12) and (4.1). If $u_1 = s$, then $u_1 = 0$ follows from (4.7) using the first equation of each of the systems (2.12). Putting $u_1 = 0$ in (4.8), we find

$$P_1 = P_2; \tag{4.12}$$

without loss of generality, using (4.2) and (4.12), we take

$$\rho_1 \le \rho_2, \quad e_1 \ge e_2. \tag{4.13}$$

In the ρ, e plane, the semiorbit of (4.11) from z_1 with ρ nondecreasing and e nonincreasing, and the semiorbit of (4.11) from z_2 with ρ nonincreasing and e nondecreasing, will each intersect the semiorbit of

$$\dot{z} = \varsigma(z) \tag{4.14}$$

from the point (ρ_1, e_2), ρ, e both increasing, where ς is obtained from (3.10). Again from strict hyperbolicity, P is a Lyapunov function on the orbits (4.14), so from (4.12) these two intersections coincide, and the above semiorbit of (4.11) from v_1 reaches v_2. □

LEMMA 4.2: *For v_1 fixed, each of the two sets $\Gamma_2^{\pm}(v_1)$ continues uniquely as a 1-manifold to infinity or to the boundary ∂D.*

Proof: Consider the system (4.9), (4.10) in the form

$$A(z_2, s)\begin{pmatrix}\dot{u}_2\\ \dot{z}_2\end{pmatrix} = \dot{s}\begin{pmatrix}z_2 - w_{1+}\\ -w_{1-}\end{pmatrix} \tag{4.15}$$

A is the same 3×3 matrix appearing in (3.2), with $u = 0$ and s replacing λ_i. Thus if s is not one of the characteristic speeds at v_2, A is nonsingular and (4.15) determines $\dot{u}_2, \dot{z}_2, \dot{s}$ up to an unimportant normalization. For $v \in \Gamma_2^{\pm}(v_1)$, having put $u_2 = 0$ in (4.9), (4.10) implies $u_1 = 0 = s; A(z_2, 0)$ is singular but the solution of (4.15) is still determined, up to normalization, if

$$\begin{pmatrix}z_2 - w_{1+}\\ -w_{1-}\end{pmatrix} = \begin{pmatrix}z_2 - z_1\\ 0\end{pmatrix} \text{ is not in the range of } A(z_2, 0). \tag{4.16}$$

From (4.9), (4.10)

$$A(z_2, 0) = \begin{pmatrix}z_2 + q_{-u}(0, z_2) & 0\\ 0 & P_z\end{pmatrix} \tag{4.17}$$

so (4.16) fails only if $z_2 - z_1$ is parallel to $z_2 + q_{-u}(0, z_2)$, both components of which are positive from (4.1), (4.2). But $\rho_1 - \rho_2, \quad e_1 - e_2$ cannot be of the same sign, as $v_2 \in \Gamma_2^{\pm}(v_1)$ implies (4.12), giving a contradiction with (4.2). □

LEMMA 4.3: *There are no solutions of (4.7), (4.8) with $z_1 \in D, \rho_2 = 0$, other than those corresponding to contact discontinuities.*

Proof: Each of the systems (2.12) includes a continuity equation for ρ. Thus in each case the first equation in (4.7) is

$$s(\rho_1 - \rho_2) = u_1\rho_1; \tag{4.18}$$

for $\rho_1 > 0, \quad \rho_2 = 0$ we have $s = u_1$ from (4.18) and thus the solution corresponds to a contact discontinuity from Lemma 4.1. □

We use these lemmas to show that the representation (4.6) holds in the large. All solutions of (4.7), (4.8) are classified as discontinuities in some field $k = 1, 2, 3$.

THEOREM 4.4: *Any solution v_1, v_2, s, of (4.7), (4.8) can be continuously deformed, satisfying (4.7), (4.8), to a trivial solution $\tilde{z}_1 = \tilde{z}_2 = \tilde{z} \in D$, $\tilde{u}_1 = 0$, $\tilde{s} = \lambda_k(\tilde{z})$. The value of k depends only on the given states v_1, v_2.*

Proof: If the given value of u_1 or of s is zero, or if $u_1 = s$, then discontinuity is a contact discontinuity, $k = 2$, and there is nothing to prove. Otherwise, as v_1, v_2 are deformed, s can never become equal to zero or to u_1, from Lemma 4.2.

At each stage we specify v_1 and solve (4.7), (4.8) for s, z_2. We reduce $|u_1|$ to zero monotonically; ρ_1 can be taken fixed; and e_1 is either fixed or increases boundedly, as explained below.

Writing (4.7), (4.8) as a system $T(z_2, s, v_1) = 0$, we readily find

$$\det \frac{\partial T(z_2, s, v_1)}{\partial(z_2, s)} = s^2 w_{1-} + P_z(z_2) \cdot (u_1 w_{1+} + q_{1-}) \tag{4.19}$$

which does not vanish for $u_1 \neq 0$, $s \neq 0$, $z_1, z_2 \in D$, so the deformation can be continued at each stage.

Next we show that $|s|, |z_2|$ are bounded as $|u_1|$ decreases and e_1 at most increases boundedly. From (4.7), (4.8), we have $|s(z_2 - w_{1+})|$ and $|P_2 + s w_{1-}|$ bounded uniformly.

If $|s| \to 0$ and $|z_2| \to \infty$ at some nonzero value of u_1, both components of the right side of (4.7) are not zero, so both components of z_2 must increase without bound. From (4.3), this implies $P_2 \to \infty$, giving a contradiction with (4.8).

Next suppose that $|s| \to 0$ and $|z_2| \to \infty$ as $u_1 \to 0$. From (4.7) we have

$$|s| = \mathcal{O}(|u_1|/|z_2|), \tag{4.20}$$

and from (4.8), P_2 remains bounded. Again, appealing to (4.3), there exists $\xi \in \mathbf{R}^2$, possibly depending on u_1, such that each component of ξ is nonnegative, $|\xi|$ and $\xi \cdot z_2$ are uniformly bounded, and $\xi \cdot (z_2 - w_{1+}) = 1$. Taking the inner product of (4.7) with ξ, the left side is $-s$ but the right side $\mathcal{O}(|u_1|)$, giving a contradiction with (4.20).

Finally, suppose that $|z_2 - w_{1+}| \to 0$ and $|s| \to \infty$. From (4.8), this could only happen as $|u_1| \to 0$. Then in (4.7), $w_{1+} = z_1 + \mathcal{O}(u_1^2)$ so

$$|z_1 - z_2| = \mathcal{O}(u_1^2 + |u_1|/|s|). \tag{4.21}$$

Then using (4.21), the right side of (4.8) is $\mathcal{O}(u_1^2 + |z_1 - z_2|) = \mathcal{O}(u_1^2 + |u_1|/|s|)$, whereas the left side is at least $\mathcal{O}(|u_1 s|)$ so $|s|$ is bounded.

Next we show that s, not zero initially, is uniformly bounded away from zero, in particular that s does not approach zero as u_1 does. Suppose otherwise, then in the limit we have $s = u_1 = 0, P_1 = q_{1+} = P_2$, so the two limits states are connected by a contact discontinuity. This is impossible, from Lemma 4.2, unless the two states obtained in the limit coincide, i.e. $|z_1 - z_2| \to 0$ in addition as $u_1 \to 0$. But for v_2 in a small neighborhood of v_1, given strict hyperbolicity, the representation (4.6) is valid [1], and s cannot be close to zero unless $s = 0$ and v_1, v_2 are connected by a contact discontinuity, which was precluded by Lemma 4.2.

Now as our deformation proceeds, since u_1, s were not originally zero it follows from Lemma 4.3 that ρ_2 cannot become zero. The same argument used to prove Lemma 4.3 shows that for the system (2.12a), (2.12e), e_2 also cannot become zero. For the other systems, however, e_2 or $e_2 + P_2$ might become zero at some point. However, in each case (2.12b), (2.12c), (2.12f) this can be avoided by a bounded increase of e_1 as $|u_1|$ is reduced, given the uniform boundedness of $|s|, |z_2|$ and the uniform boundedness of s away from zero. For each of the three systems, it suffices to write out (4.7), (4.8) and make elementary estimates. We omit these details.

Thus as $u_1 \to 0, |z_1 - z_2| \to 0$, so for $|u_1|$ sufficiently small the local representation (4.6) holds, the $\Gamma_k^{\pm}(v_1)$ are distinct, and the value of k follows from strict hyperbolicity. By continuity, the integer k is independent of the deformation, and depends only on v_1, v_2. □

In the large, we identify $\Gamma_k^+(v_1) \cup \Gamma_k^-(v_1)$ as the set of points v_2 associated with the value k in Theorem 4.4. Each of the $\Gamma_k^{\pm}(v_1)$ includes the continuation of the respective set in a neighborhood of v_1.

It is possible that $\Gamma_k^+(v_1)$ and $\Gamma_k^-(v_1)$ coincide. Because of the possible need to raise e_1 in the proof of Theorem 4.4, it is also possible that $\Gamma_k(v_1) = \Gamma_k^+(v_1) \cup \Gamma_k^-(v_1) \cup \{v_1\}$ contains disjoint segments for the systems (2.12b), (2.12c), (2.12f). If so, because of the boundedness of z_2 in proving Theorem 4.4, each disjoint segment of $\Gamma_k(v_1)$ must have (at least) one endpoint v_2 with $z_2 \in \partial D$, i.e. there must be at least two points $v_2 \in \Gamma_k(v_1)$, $z_2 \in \partial D$. A simple condition precluding this is the following.

LEMMA 4.5: *In addition to (4.2), (4.3), assume the given equation of state $P = P(\rho, e)$ is such that $P = 0$ if and only if $e = 0$. Then for $z_1 \in D$, the $\Gamma_k(v_1)$ are all connected.*

Proof: For $k = 2$, or for systems (2.12a), (2.12e), this is already established. For the other systems, we solve (4.7), (4.8) for u_1, s, ρ_2 with $P_2 = e_2 = 0$, i.e. $z_2 \in \partial D$. For (2.12c), the only solutions correspond to $u_1 = s$, or $s = 0$, i.e. contact discontinuities. For (2.12b), (2.12f), there are two such solutions which do not correspond to contact discontinuities, obtained from each other by the transformation $\rho_2 \to \rho_2, \quad u_1 \to -u_1, \quad s \to -s$. This is the reflection transformation (2.2), under which $\lambda_k(v_1) \to \lambda_{4-k}(-u_1, z_1)$ and so $k \to 4-k$ in Theorem 4.4. Thus one of the solutions $v_2, z_2 \in \partial D$ corresponds to $k = 1$ and one to $k = 3$ in Theorem 4.4; for fixed k, $\; k \neq 2, \Gamma_k(v_1)$ therefore contains at most one point v_2 with $z_2 \in \partial D$, and our result follows. □

LEMMA 4.6: *For v_1 fixed, $z_1 \in D$, assume $\Gamma_k(v_1)$ is connected, and that for some $v_2 \in \Gamma_k(v_1)$, $\; s(v_1, v_2) = \lambda_j(v_2)$. Then $j = k$.*

Remark: When genuine nonlinearity does not hold, this is a central condition in the study of the solvability of Riemann problems, for example. This question has been discussed extensively by T. P. Liu [3,4].

Proof: For $k = 2$, this is obvious from Lemma 4.1. For $k = 1$ or 3, $\Gamma_k(v_1)$ connected, $s(v_1, v_2)$ is close to $\lambda_k(v_1) = u_1 \pm c(z_1)$ for v_2 near v_1. By continuity, using Lemma 4.2, along $\Gamma_k^{\pm}(v_1)$, $\; s(v_1, v_2)$ can never become equal to u_2, by Lemma 4.2, so by strict hyperbolicity, if $s(v_1, v_2)$ becomes equal to $\lambda_j(v_2)$, $\; j$ must be k. □

The main result of this section is the following.

THEOREM 4.7: *Suppose that for all $z \in D$, $\; k = 1, 3$, $\; \Gamma_k(v)$ is connected and field k is locally genuinely nonlinear* [1], *i.e.*

$$r_k(z) \cdot \nabla_v \lambda_k(z) > 0, \tag{4.22}$$

with r_k, λ_k as given in (3.6); then each of the following holds:

(4.23) *Field k is genuinely nonlinear in the large* [6], *i.e. for all $v_2 \in \Gamma_k(v_1), v_2 \neq v_1, s(v_1, v_2) \neq \lambda_j(v_2)$ for any j.*

(4.24) *$\Gamma_k(v_1)$ continues in both directions as a smooth 1-manifold, to infinity or to $\partial D; u, P$, and $s(\cdot, v_1)$ are all strictly monotone on $\Gamma_k(v_1)$.*

(4.25) *Let $U = U(z), F = uU$ be an entropy function/flux pair, with U strictly convex in z (convex, not necessarily strictly so, in w).*

Then for $v_2 \in \Gamma_k(v_1)$ the entropy conditions,

$$\lambda_k(v_1) < s(v_1, v_2) < \lambda_k(v_2), \tag{4.26}$$

$$(4.27) \qquad -s(v_1,v_2)(U(z_2)-U(z_1))+u_2U(z_2)-u_1U(z_1)>0,$$

$$(4.28) \qquad u_2>u_1,$$

are equivalent.

(4.29) *For* $v_2 \in \Gamma_k(v_1)$ *satisfying the entropy condition, the vectors* $w_v(v_2)r_j(v_2)$, $j<k$; $w(v_2)-w(v_1)$; $w_v(v_1)r_m(v_1), m>k$ *are linearly independent.*

Remarks: From (4.23), a similarity solution of a given Riemann problem, satisfying the entropy condition, will consist of a single 1-wave (either a rarefaction wave or a shock), a contact discontinuity, and a single 3-wave. From (4.24), u, P are monotone along the curve of admissible 1-, 3-waves through any given point. Thus from Lemma 4.1, an elementary phase plane analysis in the u, P plane shows that the similarity solution of a given Riemann problem, satisfying the entropy condition, is unique in the large (assuming it exists, cf. [9]).

The result (4.29) is a statement of the stability of entropy shocks with regard to perturbations on the incoming characteristics. Some recent results to this effect [5, 7] depend critically on this assumption.

Proof: Fix v_1, with $z_1 \in D$. For definiteness, we take the case $k=3$, and consider the branch $\Gamma_3^+(v_1)$ along which λ_3 and $s(v_1,\cdot)$ are increasing initially as one moves away from v_1. Initially, $\dot v$ is parallel to r_3 given by (3.6), so $\dot u>0$, and from strict hyperbolicity, using (3.9), $\dot P>0$ as well.

Therefore if (4.23) or (4.24) is to fail, $\dot s$ and/or $\dot u$ and/or $\dot P$ must become zero at some point $v_2 \in \Gamma_3^+(v_1)$, i.e. satisfying (4.7), (4.8) and (4.9), (4.10). Let $v_2=(0,z_2)$ be the first such point in $\Gamma_3^+(v_1)$. Since u and s were increasing along Γ_3^+ between v_1 and v_2, we have

$$(4.30) \qquad u_1<0 \quad \text{and} \quad s>0.$$

From (4.9), $\dot u$ and $\dot s$ cannot vanish simultaneously, for then $\dot z=0, \quad \dot v=0$ which is not permitted by a normalization condition.

From (4.10), using (4.30), $\dot P(v_2)=0$ requires $\dot u(v_2)$ and $\dot s(v_2)$ nonpositive, which is impossible. Next suppose $\dot u(v_2)=0$ with $\dot s(v_2)>0, \quad \dot P(v_2)>0$. Using (4.9) and (4.7) we find

$$(4.31) \qquad \begin{aligned} \dot P(v_2) &= P_z(z_2)\dot z_2 \\ &= -\frac{\dot s(v_2)}{s}P_z(z_2)\cdot(z_2-w_{1+}) \\ &= +\frac{\dot s(v_2)}{s^2}P_z(z_2)\cdot(u_1w_+(v_1)+q_-(v_1)). \end{aligned}$$

From (4.30), both components of $u_1 w_+(v_1) + q_-(v_1)$ are negative, while those of P_z are nonnegative, at least one positive. Therefore the right side of (4.31) is negative, contradicting $\dot{P}(v_2) > 0$.

The remaining possibility is $\dot{s}(v_2) = 0$, with $\dot{u}(v_2) > 0$. Since $\dot{z}(v_2) \neq 0$, $s(v_1, v_2)$ has to be one of the characteristic speeds at v_2. By Lemma 4.6, $s(v_1, v_2) = \lambda_3(v_2)$, so $\dot{v}(v_2)$ is either parallel or antiparallel to $r_3(v_2)$. Because $\dot{u}(v_2) > 0$, $\dot{v}(v_2)$ and $r_3(v_2)$ are parallel, and so $\dot{\lambda}_3(v_2) > 0$ from (4.22), i.e. $\lambda_3 - s(v_1, \cdot)$ is increasing in a neighborhood of v_2. This is impossible, as from (4.22) $\lambda_3 - s(v_1, \cdot)$ is positive on $\Gamma_3^+(v_1)$ between v_1 and v_2, becoming equal to zero at v_2.

Thus (4.23) and (4.24) hold, $\lambda_3 - s(\cdot, v_1)$ is positive on all of $\Gamma_3^+(v_1)$ and $s(\cdot, v_1) > \lambda_3(v_1)$. Thus the entropy condition (4.26) is satisfied for all $v_2 \in \Gamma_3^+(v_1)$ but is violated for $v_2 \in \Gamma_3^-(v_1)$. The equivalence of the condition (4.28) follows from (4.30).

For systems which are genuinely nonlinear in the large, the entropy inequality (4.27) is equivalent to the classical entropy condition (4.26), provided U is strictly convex in w. For U convex in w, strictly convex in z, the only way (4.27) could fail to be equivalent to (4.16) is if $z_2 = z_1$ for some $v_2 \in \Gamma_3^+(v_1)$, cf. [8], eqs. (2.4), (2.5). This is precluded by the monotone increase of P on $\Gamma_3^+(v_1)$.

It remains to prove (4.29). The cases $k = 1, 3$ are entirely similar, so again we discuss the case $k = 3$. For $v_2 \in \Gamma_3^+(v_1)$, satisfying (4.7) - (4.10), we wish to show that $w_v r_1(v_2)$, $w_v r_2(v_2)$, and $w(v_1) - w(v_2)$ are linearly independent. Using (2.11), (3.3), (3.6) and $u_2 = 0$, an elementary calculation shows that linear independence will fail if and only if

$$W \equiv \begin{pmatrix} P_z^T(z_2) \\ c(z_2) \end{pmatrix} \cdot (w(v_1) - w(v_2)) = 0. \tag{4.32}$$

From (4.9), (4.10)

$$\dot{s}(w(v_2) - w(v_1)) = \begin{pmatrix} (z_2 + q_{-u}(0, z_2))\dot{u}_2 - s\dot{z}_2 \\ \dot{P}(z_2) - s w_{-u}(0, z_2)\dot{u}_2 \end{pmatrix}; \tag{4.33}$$

given genuine nonlinearity, we have already shown that $\dot{s}, \dot{u}_2, \dot{P}$ are nonzero, indeed that they are all positive. Using (4.33) in (4.32), we readily find, using (3.9),

$$W = (1 - \frac{c(z_2)}{s})(\dot{P}(z_2) + w_{-u}(0, z_2)c(z_2)\dot{u}_2), \tag{4.34}$$

which is negative, as we proved $s = s(v_1, v_2) < \lambda_3(v_2) = u_2 + c(z_2) = c(z_2)$ above, in the proof of genuine nonlinearity in the large. Thus the proof is complete. □

We note that for $k = 2$, $v_2 \in \Gamma_2(u_1)$, i.e. v_1, v_2 connected by a contact discontinuity, the vectors $w_v r_1(v_1)$, $w(v_2) - w(v_1)$, and $w_v r_3(v_2)$ are linearly independent, irrespective of genuine nonlinearity or an entropy condition. The elementary computation, which we omit, depends only on $u_1 = u_2, P_1 = P_2$, and $\rho_1/e_1 \neq \rho_2/e_2$, as obtained, for example, from (4.2).

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